

The Reversed q -Exponential Functional Relation.

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Abstract

After obtaining some useful identities, we prove an additional functional relation for q exponentials with reversed order of multiplication, as well as the well known direct one in a completely rigorous manner.

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1 Introduction

One of the most appealing results to come out of q -analysis is that the q -exponential function, defined by ${}_qD_x \exp_q(x) = \exp_q(x)$, where ${}_qD_x$ is the q -derivative, also satisfies the same defining functional relationship for ordinary exponential functions (up to normalization), given by

$$F(x)F(y) = F(x + y), \quad (1)$$

provided that $xy = q^{-1}yx$. (that is, (x, y) belongs to the Manin quantum plane.) This result was first found by Schützenberger [1] long before the non-commutative aspects of q -analysis were generally recognised and has been rediscovered many times subsequently e.g. in [2, 3]. It can be proved by means of q -combinatorics [1, 2], or by an argument based on the definition of the q -exponential as an eigenfunction of the q -derivative [3].

Besides the above well-known result, there is, in fact, an additional functional relation in the opposite order for the q -exponential functions, which is not so well known given by

$$F(y)F(x) = F(x + y + (1 - q^{-1})yx), \quad (2)$$

provided that the same condition $xy = q^{-1}yx$ holds. We first became aware of this relationship in the work of L. Faddeev and A. Yu Volkov in their study of lattice Virasoro algebra [4] who obtained a similar result in the case of a different realisation of the q -exponential, in terms of an infinite product. Their definition of the q -exponential suffered from the drawback that it did not go over into the ordinary exponential function in the commuting limit $q \rightarrow 1$.) In this paper, we will provide a completely rigorous proof of the reverse functional relation in the form stated in (2). The proof is somewhat tricky in that a seemingly unrelated identity has to be obtained first as an intermediate step.

2 Proof of the Reversed q -Exponential Functional Relation

For completeness we quickly review Schützenberger and Cigler's result, which will be used in our subsequent proof:

$$\exp_q x \exp_q y = \exp_q(x + y) \quad \text{if } xy = q^{-1}yx, \quad (3)$$

where

$$\exp_q x \equiv \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \quad [n] \equiv \sum_{k=0}^{n-1} q^k \quad [n]! \equiv [n][n-1] \cdots [1].$$

\langle Proof \rangle

$$\begin{aligned}
\exp_q x \exp_q y &= \left(\sum_{m=0}^{\infty} \frac{x^m}{[m]!} \right) \left(\sum_{n=0}^{\infty} \frac{y^n}{[n]!} \right) \\
&= \sum_{k=0}^{\infty} \sum_{r=0}^k \frac{x^r y^{k-r}}{[r]! [k-r]!} \\
&= \sum_{k=0}^{\infty} \frac{1}{[k]!} \left(\sum_{r=0}^k \frac{[k]!}{[r]! [k-r]!} x^r y^{k-r} \right) \\
&= \sum_{k=0}^{\infty} \frac{(x+y)^k}{[k]!} \quad (\text{by (A1), see Appendix}) \\
&= \exp_q(x+y) \\
&\text{Q.E.D.} \quad \square
\end{aligned}$$

Now let us go on to prove the following formula,

$$\begin{aligned}
x^n &= 1 + \sum_{r=1}^n \frac{[(q^{n-r+1} - 1)(q^{n-r+2} - 1) \cdots (q^n - 1)][(x-1)(x-q) \cdots (x-q^{r-1})]}{(q-1)(q^2-1) \cdots (q^r-1)} \\
&\equiv \sum_{r=0}^n \frac{[(q^{n-r+1} - 1)(q^{n-r+2} - 1) \cdots (q^n - 1)][(x-1)(x-q) \cdots (x-q^{r-1})]}{(q-1)(q^2-1) \cdots (q^r-1)}. \quad (4)
\end{aligned}$$

⟨ Proof ⟩

Suppose for some $n = k$, we have

$$x^k = \sum_{r=0}^k \frac{[(q^{k-r+1} - 1)(q^{k-r+2} - 1) \cdots (q^k - 1)][(x-1)(x-q) \cdots (x-q^{r-1})]}{(q-1)(q^2-1) \cdots (q^r-1)}.$$

Now, consider x^{k+1} ,

$$\begin{aligned}
x^{k+1} &= \sum_{r=0}^k \frac{[(q^{k-r+1} - 1)(q^{k-r+2} - 1) \cdots (q^k - 1)][(x-1)(x-q) \cdots (x-q^r)]}{(q-1)(q^2-1) \cdots (q^r-1)} + \\
&\quad \sum_{r=0}^k \frac{q^r [(q^{k-r+1} - 1)(q^{k-r+2} - 1) \cdots (q^k - 1)][(x-1)(x-q) \cdots (x-q^{r-1})]}{(q-1)(q^2-1) \cdots (q^r-1)} \\
&= (x-1)(x-q) \cdots (x-q^k) + \\
&\quad \sum_{r=0}^{k-1} \frac{[(q^{k-r+1} - 1)(q^{k-r+2} - 1) \cdots (q^k - 1)][(x-1)(x-q) \cdots (x-q^r)]}{(q-1)(q^2-1) \cdots (q^r-1)} + \\
&\quad \sum_{r=0}^k \frac{q^r [(q^{k-r+1} - 1)(q^{k-r+2} - 1) \cdots (q^k - 1)][(x-1)(x-q) \cdots (x-q^{r-1})]}{(q-1)(q^2-1) \cdots (q^r-1)}
\end{aligned}$$

$$\begin{aligned}
&= (x-1)(x-q) \cdots (x-q^k) + \\
&\quad \sum_{r=1}^k \frac{[(q^{k-r+2}-1)(q^{k-r+3}-1) \cdots (q^k-1)][(x-1)(x-q) \cdots (x-q^{r-1})]}{(q-1)(q^2-1) \cdots (q^{r-1}-1)} + \\
&\quad \sum_{r=0}^k \frac{q^r [(q^{k-r+1}-1)(q^{k-r+2}-1) \cdots (q^k-1)][(x-1)(x-q) \cdots (x-q^{r-1})]}{(q-1)(q^2-1) \cdots (q^r-1)} \\
&= \sum_{r=0}^{k+1} \frac{[(q^{(k+1)-r+1}-1)(q^{(k+1)-r+2}-1) \cdots (q^{k+1}-1)][(x-1)(x-q) \cdots (x-q^{r-1})]}{(q-1)(q^2-1) \cdots (q^r-1)}.
\end{aligned}$$

Since for $n = 1$, obviously we have

$$x = \sum_{r=0}^1 \frac{[(q^{1-r+1}-1)(q^{1-r+2}-1) \cdots (q^1-1)][(x-1)(x-q) \cdots (x-q^{r-1})]}{(q-1)(q^2-1) \cdots (q^r-1)},$$

the proof is complete. \square

There follows another identity which is a simple consequence of the previous one;

$$\sum_{r=0}^{m \text{ or } n} \frac{q^{\frac{r(r-1)}{2}-mn}(q-1)^r}{[m-r]![n-r]![r]} = \frac{1}{[m]![n]!}. \quad (5)$$

$\langle \text{Proof} \rangle$

$$\begin{aligned}
&\sum_{r=0}^n \frac{q^{\frac{r(r-1)}{2}-mn}(q-1)^r}{[m-r]![n-r]![r]} \\
&= \sum_{r=0}^n \frac{q^{\frac{r(r-1)}{2}-mn}(q-1)^r ([m-r+1][m-r+2] \cdots [m])([n-r+1][n-r+2] \cdots [n])}{[m]![n]![r]} \\
&= \sum_{r=0}^n \frac{q^{\frac{r(r-1)}{2}-mn} [(q^{m-r+1}-1)(q^{m-r+2}-1) \cdots (q^m-1)][(q^{n-r+1}-1)(q^{n-r+2}-1) \cdots (q^n-1)]}{[m]![n]!(q-1)(q^2-1) \cdots (q^r-1)} \\
&= \frac{1}{[m]![n]!} \sum_{r=0}^n \frac{[(q^m-1)(q^m-q) \cdots (q^m-q^{r-1})][(q^{n-r+1}-1)(q^{n-r+2}-1) \cdots (q^n-1)]}{(q^m)^n [(q-1)(q^2-1) \cdots (q^r-1)]} \\
&= \frac{1}{[m]![n]!} \quad (\text{by identity (4)}).
\end{aligned}$$

The proof is completed by noting that the above identity is symmetric in m and n . \square

Equipped with the above identity, we are now able to achieve the desired result,

$$\exp_q y \exp_q x = \exp_q [x + y + (1 - q^{-1})yx] \quad \text{if } xy = q^{-1}yx. \quad (6)$$

⟨ Proof ⟩

$$\begin{aligned}
& \exp_q y \exp_q x \\
&= \left(\sum_{m=0}^{\infty} \frac{y^m}{[m]!} \right) \left(\sum_{n=0}^{\infty} \frac{x^n}{[n]!} \right) \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} y^m x^n \sum_{r=0}^{\min\{m,n\}} \frac{q^{\frac{r(r-1)}{2} - mn} (q-1)^r}{[m-r]! [n-r]! [r]!} \quad (\text{by identity (5)}) \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\min\{m,n\}} \frac{q^{-r(n-r)} y^r x^{n-r}}{[n-r]!} \cdot \frac{q^{-\frac{r(r-1)}{2}} (1-q^{-1})^r x^r}{[r]!} \cdot \frac{y^{m-r}}{[m-r]!} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\min\{m,n\}} \frac{x^{n-r}}{[n-r]!} \cdot \frac{q^{-\frac{r(r-1)}{2}} (1-q^{-1})^r y^r x^r}{[r]!} \cdot \frac{y^{m-r}}{[m-r]!} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\min\{m,n\}} \frac{x^{n-r}}{[n-r]!} \cdot \frac{(1-q^{-1})^r (yx)^r}{[r]!} \cdot \frac{y^{m-r}}{[m-r]!} \\
&= \left(\sum_{l=0}^{\infty} \frac{x^l}{[l]!} \right) \left(\sum_{k=0}^{\infty} \frac{[(1-q^{-1})yx]^k}{[k]!} \right) \left(\sum_{h=0}^{\infty} \frac{y^h}{[h]!} \right) \\
&= \exp_q x \cdot \exp_q [(1-q^{-1})yx] \cdot \exp_q y \\
&= \exp_q [x + (1-q^{-1})yx] \cdot \exp_q y \quad (\text{by (3), as } x(1-q^{-1})yx = q^{-1}(1-q^{-1})yxx) \\
&= \exp_q [x + (1-q^{-1})yx + y] \\
&\quad (\text{by (3), as } [x + (1-q^{-1})yx]y = q^{-1}y[x + (1-q^{-1})yx]) \\
&= \exp_q [x + y + (1-q^{-1})yx] \\
&\quad \text{Q.E.D.} \quad \square
\end{aligned}$$

Appendix

The following is the so-called q -binomial expansion formula and its proof:

$$(x + y)^n = \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} x^r y^{n-r}, \quad (\text{A1})$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} \equiv \frac{[n]!}{[r]! [n-r]!} \quad [n] \equiv \sum_{k=0}^{n-1} q^k \quad [0]! \equiv 1,$$

subject to the condition that $xy = q^{-1}yx$, q being some complex number.

⟨ Proof ⟩

Suppose for some $n = k$, we have

$$(x + y)^k = \sum_{r=0}^k \begin{bmatrix} k \\ r \end{bmatrix} x^r y^{k-r}.$$

Now consider $(x + y)^{k+1}$,

$$\begin{aligned} (x + y)^{k+1} &= \sum_{r=0}^k \frac{[k]!}{[r]![k-r]!} (x + y) x^r y^{k-r} \\ &= \sum_{r=0}^k \frac{[k]!}{[r]![k-r]!} x^{r+1} y^{k-r} + \sum_{r=0}^k \frac{[k]!}{[r]![k-r]!} q^r x^r y^{k-r+1} \\ &= x^{k+1} + \sum_{r=0}^{k-1} \frac{[k]!}{[r]![k-r]!} x^{r+1} y^{k-r} + \sum_{r=0}^k \frac{[k]!}{[r]![k-r]!} q^r x^r y^{k-r+1} \\ &= x^{k+1} + \sum_{r=1}^k \frac{[k]!}{[r-1]![k-r+1]!} x^r y^{k-r+1} + \sum_{r=0}^k \frac{[k]!}{[r]![k-r]!} q^r x^r y^{k-r+1} \\ &= x^{k+1} + \sum_{r=1}^k \frac{[k]!(1 + q + \cdots + q^{r-1})}{[r]![k-r]!(1 + q + \cdots + q^{k-r})} x^r y^{k-r+1} + \\ &\quad \sum_{r=0}^k \frac{[k]!}{[r]![k-r]!} q^r x^r y^{k-r+1} \\ &= x^{k+1} + \sum_{r=0}^k \frac{[k]!(1 + q + \cdots + q^k)}{[r]![k-r]!(1 + q + \cdots + q^{k-r})} x^r y^{k-r+1} \\ &= \sum_{r=0}^{k+1} \frac{[k+1]!}{[r]![k+1-r]!} x^r y^{k+1-r}, \end{aligned}$$

so the same formula holds for $n = k + 1$.

Since for $n = 1$, obviously we have $x + y = \sum_{r=0}^1 \frac{[1]!}{[r]![1-r]!} x^r y^{1-r}$, the proof is complete. \square

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